Description of the experimented study process

The study process has been experimented for 11 sessions of 50 minutes, in a class of 30 students of grade 7 (12-13 years old) of Madrid. The study process has been addressed by a researcher, Tomás Sierra, accompanied by the usual teacher of the class. Below we present a schematic description on the mathematical activity developed and the main issues addressed at different moments of the study process.

We have included the reference[1] to the page in red, inside this document, where the concrete material[2] used in every session is situated.

Session	*Task sequence and main moments of the study process*	*Issues to notice*
1	- *Explanation of the study conditions: working groups, homework, assessment…* - *Presenting Q_0* - *First encounter with the numeration problem and the existence of multiples SN (M p. 4)* - *Approaching Q_1 - Q_6 in the case of the Roman NS (homework, from now on HW) (M p. 5)*	*Examples of writing proposals:* *Additive: 10+7+8+11+7, 8+8+8+8+8+3* *Multiplicative-additive: $6\times 7+1$, $8\times 5+3$, forty-three, $4\times 10+3$* *Decimal positional: 43* *Multiplicative-subtractive: $6\times 8-5$* *With division: 86/2*
2	- *Debate: Roman NS' properties and comparison with the habitual positional NS.* - *First encounter with other additive NS (Egyptian) (M p. 6)* - *Approaching Q_1, Q_2, Q_3 ,Q_4 (HW)*	*Clarification of the rules of writing in the Roman NS and the different use of position with respect to the habitual positional NS.*
3	- *Debate: Egyptian NS' properties and comparison with the habitual positional NS.* - *Exploration of MP_a: calculations (HW) (M p. 7)*	*The students present their answers to Q_1, Q_2, Q_3, Q_4 about the Egyptian NS and they are discussed in large group.*
4	- *Debate: calculations in MP_a* - *Exploration and evaluation of MP_a* - *Approaching Q_5 - Q_{12}* - *Explanation of the Egyptian multiplication technique* - *Search of the Egyptian division technique (HW) (M p. 8)*	*Limitations of the additive NS. Students intend to create new symbols or to group them to represent large numbers.* *For the multiplication and division, students make new inefficient techniques. For the division, they covert all the symbols in "strokes" and then group them.*
5	- *Work with the division technique, evaluating and institutionalizing the work done (M pp. 9 a 12)* - *Explanation of the Egyptian division technique*	*In the answer to Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}, the limitations of the additive NS appear. The Egyptian multiplication and division techniques are practiced. No need*

[1] For the identification of the materials we use the expression: M p. x (Material situated in page x).

[2] The original materials (in Spanish) can be consulted in Sierra (2006, pp. 439-472).

	- *Analyze of additive NS' limitations (HW)*	*for multiplication tables.*
6	- *Debate: additive NS' limitations, analyzing the economy, reliability and domain of validity of MP_a. Comparison with the habitual positional NS.* - *Theoretical work, evaluation and institutionalization of MP_a. First encounter with MP_h (M p. 13)* - *Exploration of MP_h (Chinese NS) (HW)*	*In the practice of calculations in MP_a and in the answer to Q_{12}, a student propose the following: writing BBBBB AAAAAAA IIIIIIII like (IIIII)B (IIIIIII)A (IIIIIIII)I (where B=100, A=10 e I=1). This causes the first encounter with MP_h.*
7	- *Debate: Exploration of MP_h* - *Approaching Q_1-Q_{12} on the Chinese NS.* - *Debate: Chinese NS' properties.* - *Debate: comparison between number writings in MP_a and in MP_h.* - *Construct a chart about the comparison of number writings in the "hybrid" NS from other one about additive NS.* - *Working the techniques of MP_h (HW) (M p. 14)*	*In the multiplication, some students propose to do it like with polynomials, e.g.: 6A 7 $\times$ 3A 8 is done 8$\times$7, 6A$\times$3A, 6A$\times$8, 7$\times$3B, and for calculating 6A$\times$3A is necessary multiply 6$\times$3 = A 8 y A$\times$A = B and A 8$\times$B = C 8B (where A=10 y B=100).* *Some students propose the use of the multiplication tables not only with the coefficients but also with the powers of the base in order to do it faster.*
8	- *Theoretical questioning evaluation and institutionalization of MP_h.* - *Debate: assessment of MP_h (M pp. 15 a 19)* - *Analysis of the oral SN in MP_h* - *First encounter with MP_p: the Mayan NS.* - *Approaching Q1, Q3, Q4, Q13, Q14 for the Mayan NS (HK) (M p. 20) y (M pp. 21 a 25)*	*Doubts about how multiplication and division in MP_h are treated. Aims to eliminate the symbols of the powers of the base in the hybrid NS in order to improve them. Represent the symbols with the localization of their coefficient causes the first encounter with MP_p.*
9	- *Exploration of MP_p* - *Debate: comparison between Mayan NS and habitual positional NS.* - *Work of the technique and evaluation moments.* (HK) *(M pp. 26 y 27)*	*Some students propose the use of only one symbol for each coefficient (20 symbols) to avoid ambiguity. Thus, the full positional NS with base 20 arises. This helps to understand why our habitual NS has no problems of ambiguity.*
10	- *Synthesis of the constructed local MP picking up the generative question and the evolution in the search of an answer to it.* - *A dossier with a synthesis of the process of construction of the local MP is given to the students. (M pp. 28 a 32)*	*Students and teacher make a final balance of the process, pointing out advantages and limitations of each studied NS.*

11	- *Individual evaluation of the students: 6 questions. (M p. 33)*	*3 questions in which students have to apply a known technique with different data.* *1 question in which students have to apply the inverse technique to one studied.* *2 questions about theoretical justifications.*

Materials of the study process developed in secondary school

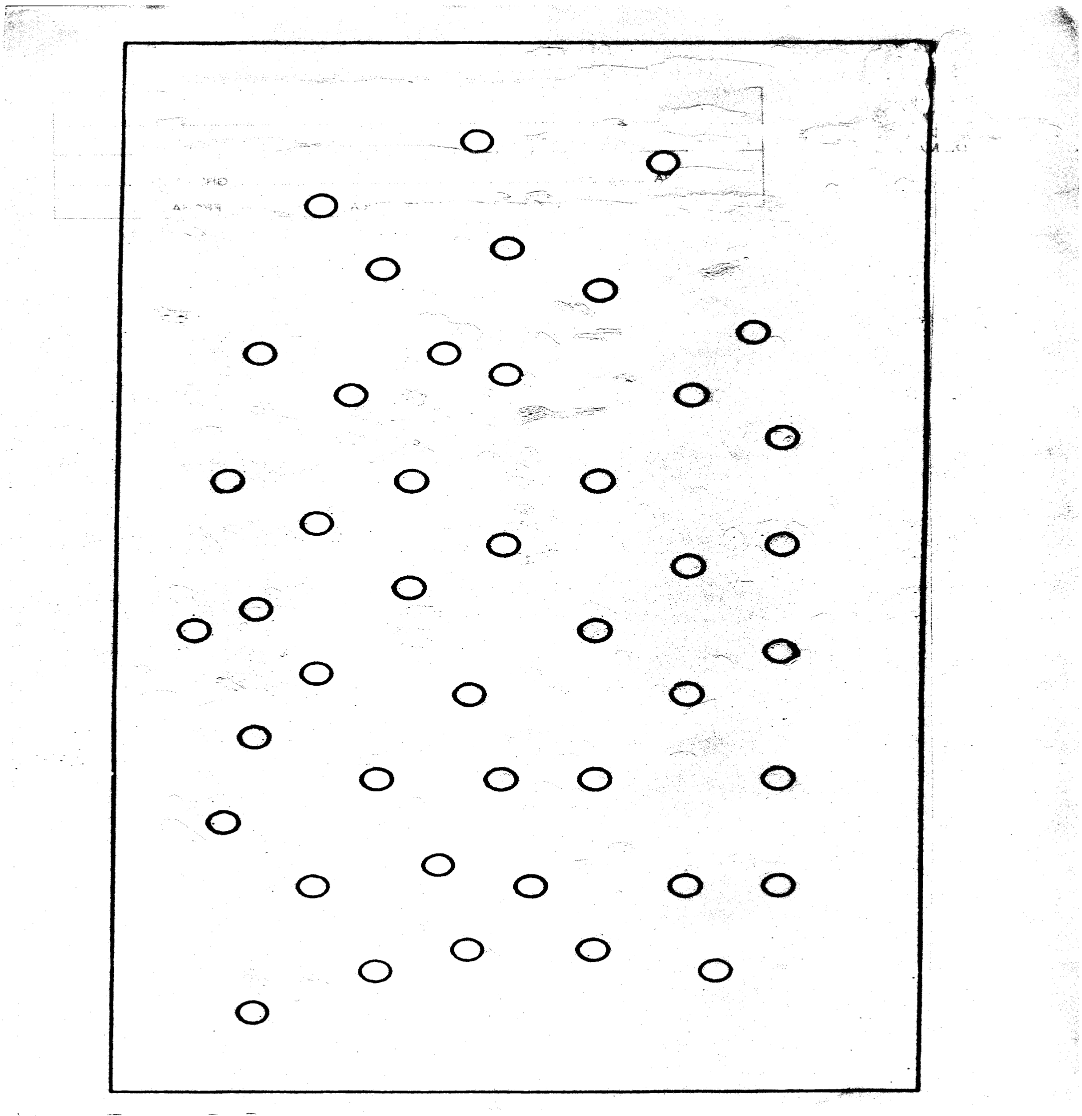
Materials for the 1st session (pp. 4 & 5)

Initial question:

"We have an issuer and a receiver. The issuer has the following collection of plates which the receiver cannot see. The issuer has to send a written message to the receiver so that he brings exactly the necessary spoons to put one in every plate."

There are several solutions to this problem.

In small groups we will start by searching for at least four different ways of issuing the above mentioned message.



Homework of the 1st session

THE ROMAN SYSTEM

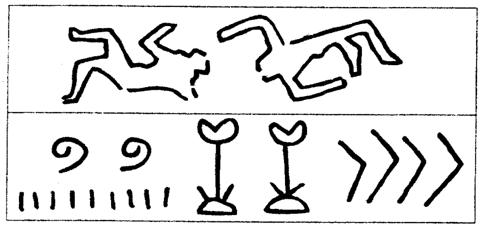
1. Write the numbers 143, 1001, 1998 and 2004 in the Roman System.
2. Add up the previous four numbers in the Roman System.
3. To emphasize three differences between our NS and the Roman System.

Materials for the 2nd session (p. 6)

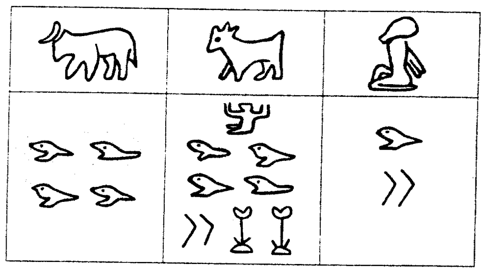
Egyptian Numeral System

In ancient Egypt, some pharaohs built temples in honor of their gods. They had the walls decorated with sculptures and paintings which showed the most glorious episodes of their lives or with daily life scenes. The three following documents have been discovered in two of the temples:

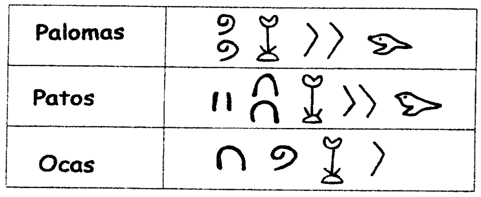
The first one indicates the number of enemies massacred in a battle which was won by the Pharaoh from HIERAKONPOLIS: 42209 men.



The second one indicates the number of men, of goats and of oxen captured in that battle:
120 000 men
1 422 000 goats
400 000 oxen



The third one indicates the number of animals a Pharaoh from Memphis had:
121 200 doves
121 022 ducks
11 110 geese



What rules were used by Egyptians to write numbers?

In this document there are several numbers written in the Egyptian NS and also, in the decimal positional NS.

How does the Egyptian system work?

(a) What group of symbols is used? Have all the symbols got the same role?
(b) What natural number is represented by each symbol?
(c) How can you know if a group of symbols represents a number or not? (That is, how can you know if a number is correctly or incorrectly written? What rules of writing are there? How is the value of a number obtained?
(d) Can the same set of symbols represent two different numbers?
(e) If we want to write very big numbers, will many symbols be necessary?
(f) If one number is bigger than another one, will more symbols be necessary to write it?

Materials for the 3rd session (p. 7) Homework

Exploratory task a_1

You have to designate the cardinal of each of the following collections by means of the Egyptian NS:

⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗⊗
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Exploratory task a_2

Supposing that the symbols easier to use that we have chosen to represent the numbers in the Egyptian system are:

$1 \rightarrow I \quad 10^1 \rightarrow A \quad 10^2 \rightarrow B \quad 10^3 \rightarrow C \quad 10^4 \rightarrow D \quad 10^5 \rightarrow E$ y $10^6 \rightarrow F$

Construct a collection which has: a) AAAIIIIIII elements,

b) AAAAAAAAAIIIIII elements.

Exploratory task a_3

Write the following numbers in the Egyptian System: 8, 51, 99, 103, 320, 895, 999, 2874, 10001, 8000100 and 25384295.

Exploratory task a_4

Order the following series of numbers (without translating them into its habitual expression) and explain the technique that you have used:

- E, CAAIIII, FFDCCAII, AAAAIIIIII, DCAI, DDDDDCCCCCCBBBAAAAIIIII, BBBBBBBBBAAAAAAAAAIIIIIIII,
- ECC, DCAAII, FBI, AAAAAAAA, AAAAAAAIIIIII, BBBBBBBAAAAAAI, DBBBBBBBBBIIIIIIII, ECBBBBBBBBAAAAAAAIIIIIIII, DDI,

Exploratory task a_5:

Do the following operations in the additive NS, and in every case, explain the used technique and then verify the result with the normal NS.

- ¬ Calculate CCCBBBBBBBBBAAAAAAIIIIIIII + CCCCCCCAAAAAAAII
- ¬ Calculate BBBBAAAIIIIIII – BAAAAAAAAAII
- ¬ Calculate AIIII × IIIII
- ¬ Divide AAAAIIIII ÷ IIIII
- ¬ Find the common divisors of AAIIII and AAA
- ¬ Find the common divisors of AAIIII and AAA

Materials for the 4th session (p. 8) Homework

Additive numeral systems

<u>Task a$_6$</u>: Do the following calculations in the additive NS and explain, in every case, the used technique. Then verify the result with the habitual NS:

¬ Calculate CAIIIIII – BBBBAAAIIIIIIII

¬ Calculate AAAAIIIIIII × AAIII

¬ Divide BBBBBAAAIIIIIII ÷ AAIIII

<u>Task a$_7$</u>: The teacher proposes to start by doing the same activities which were done in the Egyptian NS, but now in the Roman numeral system.

Questions about additive numeral systems

a) Can all the natural numbers be written in the additive NS? What is special about the Roman NS? Is it more efficient than the other additive NS?
b) Can the different arithmetic operations always be done? With what numbers are the calculations almost impracticable (*domain of validity*)? Do the peculiarities of the Roman NS enable it to be more o less effective?
c) In the additive NS are the adding and multiplication tables necessary? Why?
d) At what point does the magnitude of numbers make the operations difficult to perform without any error?
e) At what point does the magnitude of numbers make the operations too tedious and slow?
f) How is the written number modified when it is multiplied by a power of the basis?
g) What changes have to be introduced in the NS in order to improve the multiplication and division algorithms?

Materials for the 5th session (pp. 9-12)

Multiplication and division exercises in the Egyptian NS

Multiply 37 × 245 in **an “additive” system**

Successive duplications of one of both terms (here of 245) in the following way:

→ I	BB AAAA IIIII	BB AAAA IIIII ←
I I	BB AAAA A ← ~~IIIII~~ BB AAAA ~~IIIII~~	BB AAAA A BB AAAA
→ II II	BB AAAA BB B ← ~~AAAA A~~ BB ~~AAAA A~~ CC XXXX	BBBB B AAAA ← BBBB AAAA
IIII IIII	BBBB AAA B ← C ← ~~BBBBB~~ ~~AAAAA~~ ~~BBBBB~~ ~~AAAAA~~ BBBB AAA	BBBB B AAA C BBBB AAA
A ← ~~IIIII~~ III ~~IIIII~~ III	BBBB B ← CC ← ~~BBBBB~~ ~~AAAAA~~ ~~BBBBB~~ ~~AAAAA~~ C BBBB AA	CCC BBBBB AA BBBB
→ A A ~~IIIII~~ I A ← ~~IIIII~~ I	CCC BBBB AA ~~BBBBB~~ CCC C ← ~~BBBBB~~ BBBB AA	CCC BBBB AA ← CCCC BBBB AA

We can represent the previous calculations in our decimal positional NS in a much simpler way:

→ 1	245 ←
2	490
→ 4	980 ←
8	1960
16	3920
→ 32	7840 ←

To calculate 37 times 245, successive duplications of 245 are done. The process stops in 32 times 245 because the following duplication would provide 64 times 245, which overcomes 37 times 245, which is what we want to calculate. In order to complete the calculation, we look for the numbers, in the left column, that will be added to 32 to get

37 (they are 4 and 1)[3] and these two numbers are marked as well as 32 and their correspondents in the column of the right (these are 980, 245 y 7480). Finally, the marked numbers are added in the column of the right and the product is obtained:

7840 {	CCC	BBBB	AA	
	CCCC	BBBB	AA	
980 {		BBBBB	AAAA	
		BBBB	AAAA	
245 {		BB	AAAA	IIIII
9065 {	CCCCC		AAA	III
	CCCC		AAA	II

Divide 1475 ÷ 43 in an "additive" system

Duplications of 43 are done so as to get as closer as possible to 1475:

I	AAAA III	AAAA III
→ I I	AAAA III AAAA III	AAAA III ← AAAA III
II II	~~AAAAA~~ AAA ~~IIIII~~ I ~~AAAAA~~ AAA ~~IIIII~~ I B A	B AAAA II AAA
IIII IIII	B B ← ~~AAAAA~~ AA II B ~~AAAAA~~ AA II	BBB AAAA IIII
A IIIIII	BBB AAAA IIII BBB AAAA IIII	BBB AAAA IIII BBB AAAA IIII
→ AAA II	~~BBBBB~~ B ~~AAAAA~~ AAA ~~IIIII~~ III ~~BBBBB~~ B ~~AAAAA~~ AAA ~~IIIII~~ III C B A	← C BBB AAAA III AAA III

We can represent the previous calculations with our decimal positional NS in a much simpler way:

1	43
→ 2	86 ←
4	172
8	344
16	688
→ 32	1376 ←

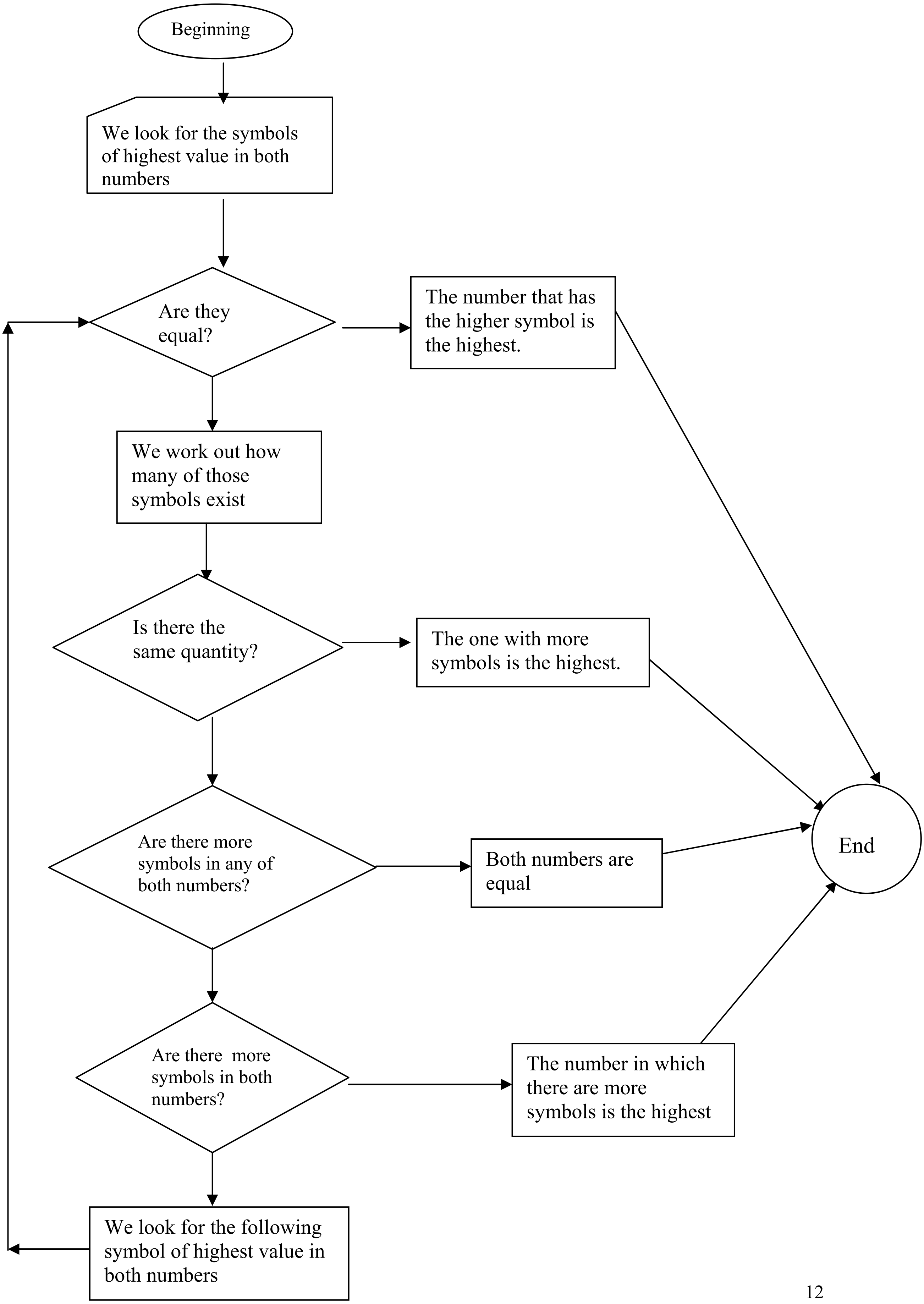
[3] It can easily be demonstrated that any natural number is the addition of any number to the power of 2....
Puede demostrarse fácilmente que cualquier número natural es suma de potencias de 2.

We stop in 1376, in the right column, because the following duplication would give a number superior to the dividend 1475. Later, we look for the numbers, in the right column, that added to 1376 get as close as possible to 1475. We get “86”: 1376 + 86 =1462

1376	C BBB AAAA III
	AAA III
	AAAA III
86	AAAA III
1462	C BBBB AAA II
	AAA

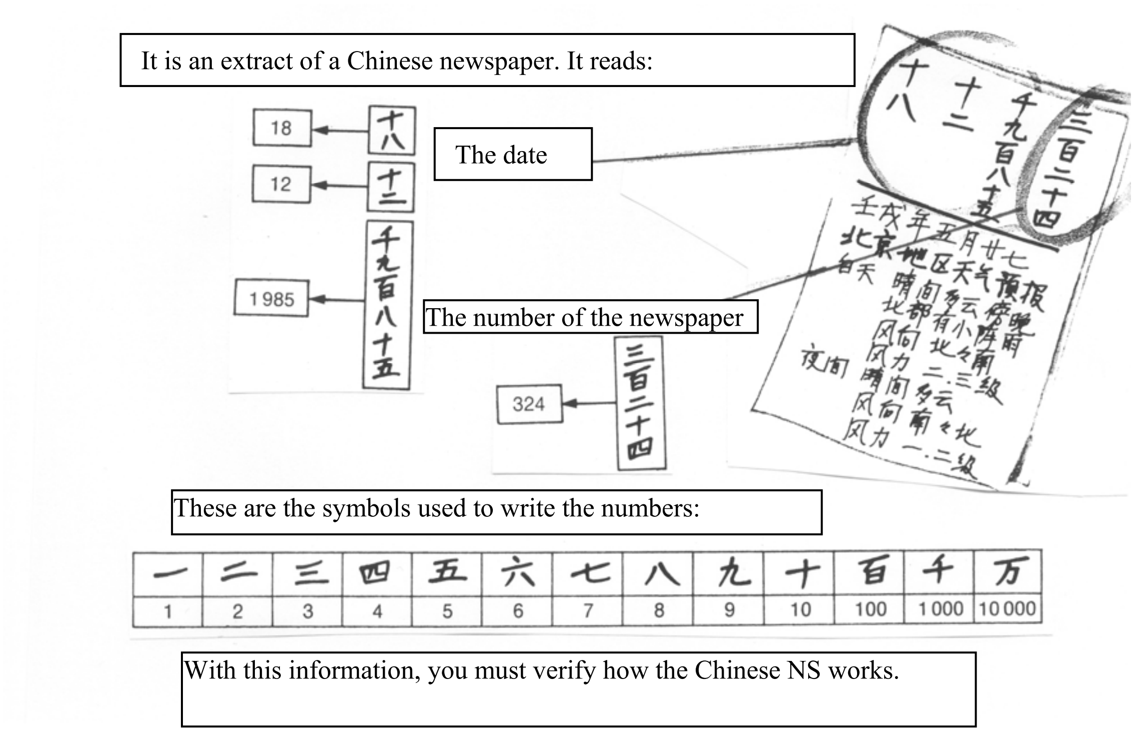
As “1462” needs “13” to reach “1475”, the *difference of the division* is “13”. And as the numbers that accompany “1376” and “86” are, respectively, “32” and “2”, it turns out that the *quotient of the division* is “34”.

Comparison of two written numbers in an "ADITIVE" system



Materials for the 6th session (p. 13)

<u>Task to manage the first encounter with the Chinese NS</u>:
In the following document there are a few written numbers in the Chinese NS as well as in the system of decimal numeration.



The following questions are suggested:

a) What natural number is represented by each symbol in the Chinese system?
b) Change the Chinese symbols into others that are easier to draw and that perform the same functions.
c) Explain how each of the symbols that appear in the Chinese NS works?
d) *What arithmetic operations correspond to the juxtaposition or adjunction of the symbols in the Chinese NS?*
e) *Is there a symbol in the NS to represent the zero?*
f) What changes appear in the Chinese NS in relation with the Egyptian NS?

Materials for the 7th session (p. 14)

Exploratory work in the “hybrid” system

Exploratory task h_1:
Designate the cardinal of each of the following collections by means of the hybrid NS:

♦♦♦♦♦♦ ♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦ ♦♦♦♦♦♦♦♦♦♦♦♦♦
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Exploratory task h_2:
Supposing that the symbols easier to use that we have chosen to represent the numbers in the Chinese system are:

$I \rightarrow 10^0$; $A \rightarrow 10^1$; $B \rightarrow 10^2$; $C \rightarrow 10^3$; $D \rightarrow 10^4$; $E \rightarrow 10^5$; $F \rightarrow 10^6$; ...

And as new symbols that will function as the multipliers of the above mentioned powers 2, 3, 4, 5, 6, 7, 8 y 9[4].

Construct a collection that has 9A 6 elements, etc.

Exploratory task h_3:
Write in the Chinese NS the numbers: 8, 99, 103, 999, 10001, 8000100 and 25384295.

Exploratory task h_4:
Order the following series of numbers (without translating them into its habitual expression) and explain the technique that you have used:

E, C 2A 4, 2F D 2C A 2, 4A 6, D C A I, 5D 6C 3B 4A 5, 9B 9A 8.
E 2C, D C 2A 2, F B I, 8A, 7A 6, 7B 6A I, D 9B 8, E C 8B 7A 9, 2D I.

Exploratory task h_5:
Do the following operations in the hybrid NS. Explain the used technique in every case and then verify the result with the normal NS:

Calculate 3C 9B 6A 8 + 7C 7A 2

Calculate 2D 9B 3A I – 8C 9B 5A 3

Calculate 4A 7 × 9A 2

Divide 2C 3B 4 ÷ 7A 9

Find the common divisors of 2A 4 and 3A

Find the common divisors of 2A 4 and 3A

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[4] The number 1 is not used as multiplier because it is not necessary.

Materials for the 8th session (pp. 15-18)

Work in the HYBRID numeration system

- **Order 5F 9D 7C 8A 9 y 5F 9D 7C 9A 8.**
- **Design an algorithm for the comparison of any two written numbers in the hybrid NS.**
- **Calculate F A I – 3E C B.**
- **Calculate 3C 7B × D 3A.**
- **Divide 2E 3D 3C 5B I by 2C 5A 7**

Questions about the HYBRID numeration system

a) Can all natural numbers be written in the hybrid NS?
b) Can all different arithmetic operations always be done? With what numbers are the calculations almost impracticable (domain of validity)?
c) In the hybrid NS are the adding and multiplication tables necessary? Why? What are these tables like?
d) At what point does the magnitude of numbers make the operations difficult to perform without any error? (reliability)
e) From what magnitude of numbers can the operations be done more rapidly? (economy)
f) How is the written number modified when it is multiplied by a power of the basis?
g) When we use the writing “nine hundred and forty five thousand two hundred and eighty three” to designate the number 945283, what kind of numeration system are we using? Analyze the characteristics of the system.
h) What changes would there be necessary to introduce in the hybrid representation technique to improve the possible techniques of multiplication and division?

Multiply 2745×389 in an “hybrid” system

To multiply 2C 7B 4A 5 by 3B 8A 9, we have to use two multiplication tables: the coefficients table and the powers of the base table.

To do the product, it is necessary to multiply each component of the first number, 3B 8A 9, by every component of the second one, 2C 7B 4A 5. We have to start by multiplying 9 by 2C, by 7B, by 4A and by 5 and we have to add the four obtained results. Later, 8A is multiplied by 2C, by 7B, by 4A, by 5 and so on. To obtain the final result, we have to add the obtained quantities in three partial multiplications.

```
     2C  7B  4A  5                        2C  7B  4A  5
 ×        3B  8A  9     ———→           ×            9     ———→
 ——————————————————                  ————————————————
                                                4A  5
                                            3B  6A
                                        6C  3B
                                     D  8C
                                    ————————————————————
                                    2D  4C  7B       5
```

```
     2C  7B  4A  5                     2C  7B  4A  5
         ×        8A                    ×    3B
 ——————————————————      ———→         ————————————————
                 4B                          C   5B
             3C  2B                       D  2C
          5D 6C                       2E  D
      E   6D                          6E
 ————————————————————                ————————————————————
   2E    D  9C  6B                    8E  2D  3C  5B
```

Then, the three partial results are added:

```
               2D    4C    7B      5
          2E    D     9C    6B
          8E   2D     3C    5B
 ————————————————————————————————————
      F        6D    7C     8B     5
```

Multiplication table of the coefficients:

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	A	A 2	A 4	A 6	A 8
3	3	6	9	A 2	A 5	A 8	2A 1	2A 4	2A 7
4	4	8	A 2	A 6	2A	2A 4	2A 8	3A 2	3A 6
5	5	A	A 5	2A	2A 5	3A	3A 5	4A	4A 5
6	6	A 2	A 8	2A 4	3A	3A 6	4A 2	4A 8	5A 4
7	7	A 4	2A 1	2A 8	3A 5	4A 2	4A 9	5A 6	6A 3
8	8	A 6	2A 4	3A 2	4A	4A 8	5A 6	6A 4	7A 2
9	9	A 8	2A 7	3A 6	4A 5	5A 4	6A 3	7A 2	8A 1

Multiplying of the powers of the base table:

×	A	B	C	D...
A	B	C	D	E
B	C	D	E	F
C	D	E	F	...
D	E	F	...	...

Divide 3589 by 74 in an "hybrid" system

```
3C  5B  8A   9      |    7A  4
2C  9B  6A          ----------
-----------            4A  8
    6B  2A   9
    5B  9A   2
-------------

        3A   7
```

In the multiplication table of the powers of the base ,we have to look for the power of the base that multiplied by A gives C and turns out to be B, B × A = C, but as the coefficient of A is 7, B × 7A = 7C, which is bigger than 3C. ,. We must choose A, which is the power of the immediately lower base. Later, the table of coefficients is used to see what coefficient corresponds to A, and we can see that it is 4A.

Thus, 4A × (7A 4) = 2C 9B 6A, which subtracted from 3C 5B 8A 9, provides the partial dividend 6B 2A 9. In the same way, we must look for what power of the base, with its corresponding coefficient, multiplied by 7A 4 gets as closer as possible to 6B 2A 9. According to the multiplication tables, the wanted number is "8". Therefore: 8 × (7A 4) = 5B 9A 2 and subtracted from 6B 2A 9 gives as difference 3A 7.
In short, the *quotient* is 4A 8 and the difference is 3A 7, because

3C 5B 8A 9 = (7A 4) × (4A 8) + 3A 7

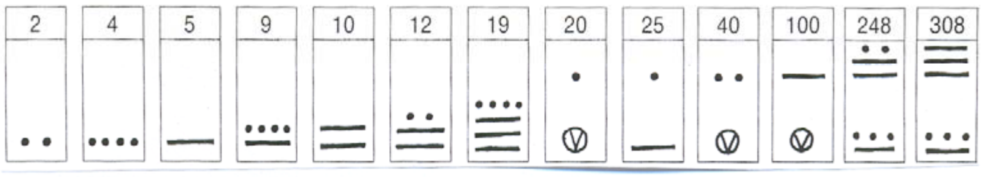
Multiplication table in the Chinese SN done by a student

×	一	二	三	四	五	六	七	八	九	十
一	一	二	三	四	五	六	七	八	九	十
二	二	四	六	八	十	十二	十四	十六	十八	二十
三	三	六	九	十二	十五	十八	二十一	二十四	二十七	三十
四	四	八	十二	十六	二十	二十四	二十八	三十二	三十六	四十
五	五	十	十五	二十	二十五	三十	三十五	四十	四十五	五十
六	六	十二	十八	二十四	三十	三十六	四十二	四十八	五十四	六十
七	七	十四	二十一	二十八	三十五	四十二	四十九	五十六	六十三	七十
八	八	十六	二十四	三十二	四十	四十八	五十六	六十四	七十二	八十
九	九	十八	二十七	三十六	四十五	五十四	六十三	七十二	八十一	九十
十	十	二十	三十	四十	五十	六十	七十	八十	九十	百

Materials for the 8th session HOMEWORK (p. 20)

The Mayan NS

Original of Central America, the Maya were very interested in astronomy. Also, they needed to write big numbers. Nevertheless, they used only three symbols.
Next, we will show several numbers written in the Mayan system.



1. Find how the Mayan NS works.
2. Try to explain why the Mayan NS allows us to write much bigger numbers than the Egyptian system.

The following questions are proposed:

a) What natural number is represented by each symbol?
b) Explain how each symbol of the Mayan NS works?
c) Can the same writing represent two different numbers? If that is the case, give an example and explain how the above mentioned problem of ambiguity could be avoided.
d) *What arithmetic operations correspond to the juxtaposition or adjunction of the symbols in the Mayan NS?*
e) What role does the position of symbols play within the group of symbols which represent a number in the Mayan NS?
f) Is *zero* used as a symbol in the Mayan system? If that is the case, what is its use?
g) What changes appear in the Mayan NS in connection with the Egyptian and Chinese NS?
h) Write the numbers 205, 999, 100000 y 25384 987 in the Mayan system.

Materials for the 8th session
So that they become familiar with the different techniques of calculation in the decimal positional NS
(pp. 21-25)

Some subtraction algorithmic techniques in the decimal positional NS

The following subtraction techniques can be used:

1. **The technique of "borrowing".**

Both numbers are aligned on the right one on top of the other and the subtraction begins from right to left, to the units of the first order, to the units of second order and so on. This technique, in case the figures of the minuend are always bigger than the corresponding figures of the subtrahend, turns into five independent subtractions, one for every position.

When this is not like that, the technique consists of transforming the writing of the minuend so that each of the values that appear in every position is bigger than each of the corresponding numbers that appear in the subtrahend.

For example, if we want to reduce 2475 – 187, we have to transform the writing of 2475 so that all the values that appear in each of the positions are bigger than the corresponding numbers of 1879. Then we have to write 2475 as 1 (13) (16) (15). Thus, we will have 15 units of the first order, 16 of the second order, 13 of the third order and 1 of the fourth order. We have transformed a unit of second order into units of the first order. Thus, we have 10+5 units of the first order and 6 units of the second order as rest. In the same way, we have transformed a unit of the third order into units of the second order. Therefore, we have 10+6 units of the second order, and we have transformed a unit of the fourth order into units of the third order, so we have 10+3 units of third order, and we have 1 unit of the fourth order left. Now we can do the subtraction position to position:

1	13	16	15
1	8	7	9
0	5	9	6

2. **Classic or the “Fibonacci” technique.**

This technique equally, consists of placing minuend and subtrahend one on top of the other, aligning them to the right and, when some figures in the minuend are smaller than the corresponding figures of the subtrahend, 10 units of that order are added to the minuend. Then, one unit of the following order is added to the subtrahend. Thus, if we add the same number to the minuend the difference does not change (since 10 units of an order are equivalent to 1 unit of the following order). For example:

2	$_{1}4$	$_{1}7$	$_{1}5$
1_{+1}	8_{+1}	7_{+1}	9
0	5	9	6

3. **The “for compensation” technique**

This technique, like in the other techniques, consists of placing minuend and subtrahend one on top of the other, aligning them to the right and, when there are some numbers in the minuend that are smaller than the corresponding figures of the subtrahend, the minuend changes into another number where the first number on the left side remains equal and all the other numbers are transformed into 9. Thus, we add a number to the minuend and, we also have to add the same number to the subtrahend so that the difference does not change. For example:

2	4	7	5	+ 524 →	2	9	9	9
1	8	7	9	+ 524 →	2	4	0	3
					0	5	9	6

4. **The “addition with hollows” technique.**

This technique consists of placing, equally, minuend and subtrahend one on top of the other, aligning them to the right and now the objective is to find what number we have to add to the subtrahend to obtain the minuend. For example:

2	4	7	5
1^1	8^1	7^1	9
0	5	9	6

It is done from right to left, position to position, as follows: we calculate what number is necessary to add to 9 to obtain 15 (9 to reach 15, 6 and I take 1), then what it is necessary to add to 7+1 to obtain 17 (from 8 to get to 17, 9 and I take 1), later what it is necessary to add to 8+1 to obtain 14 (from 9 to get to 15, 6 and I take 1) and finally what it is necessary to add to 1+1 to obtain 2.

Some algorithmic techniques for multiplication in the positional NS

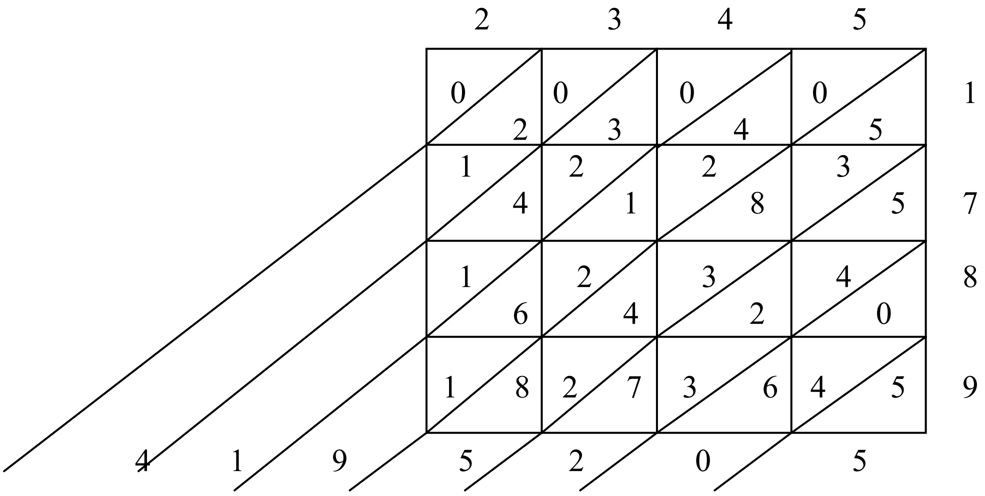
1. The “double entry” technique.

We have to place the numbers as if they were the dimensions of a rectangle, and, to do that, we do the canonical decomposition of every number and do the reduction of the writings in chunks. Then, we add the obtained result in each of the columns and, finally, add the total of every column. For example:

2000	300	40	5	
2000000	300000	40000	5000	1000
1400000	210000	28000	3500	700
160000	24000	3200	400	80
18000	2700	360	45	9

3578000+536700+ 71560 + 8945 = 4195205

2. The “per Gelosía” technique.



3. The classic technique

			2	3	4	5
		×	1	7	8	9
		2	1	1	0	5
	1	8	7	6	0	
1	6	4	1	5		
2	3	4	5			
4	1	9	5	2	0	5

Some division techniques in the positional NS

1. The Anglo-Saxon technique. (Several examples)

Divide 3589 ÷ 74 using:

	48
74	8
	40
74 × 40 = 2960	**3589**
	-2960
74 × 8 = 592	0629
	- 592
	037

The quotient is 48 y the difference is 37, because
3584 = 74 × 48 + 37 y 37 < 74.

Divide 960084 ÷ 562

562	
562 × 1	**1708**
562 × 7 = 3934	
562 × 8 = 4496	**960084**
	- 562000
	398084
	- 393400
	004684
	- 4496
	0188

Or also:

	1708
562	1 2
562 × 1	16 6
562 × 6 = 3372	
562 × 2 = 1124	**960084**
	- 562000
	398084
	- 337200
	060884
	56200
	004684
	3372
	1312
	1124
	0188

2. The “useful multiples” technique

Divide 4837 ÷ 17

First, write the following multiples of 17:

17	34	51	68	85	102	119	136	153
170	340	510	680	850	1020	1190	1360	1530
1700	3400							

Later:

4837	
- 3400	200
1437	
- 1360	80
77	
-68	4
9	284

The quotient is 284 and the difference is 9

Exercise: Explain the above algorithm and indicate what reasons there are to consider only the multiples of 17, which have been written before and not others. Apply it to the division 43257 ÷ 58.

Materials for the 9th session HOMEWORK (pp. 26 and 27)

Tasks in the decimal positional NS

1.- Order the following numbers:
123000004500321009876, 123000004500321009879,
1239987899672348213 and 1239987899672345678.
Explain the used technique. Suggest a technique in general that allows us to compare two numbers from its written form in the decimal positional NS.

2.- Calculate with three different techniques. Explain and justify the three proposed techniques:

a) 27 + 38 + 16 + 9 + 24 + 12 + 33,
b) 35073 + 38 + 2300045007 + 895000 + 5

3.- Calculate with four different techniques. Explain and justify the four proposed techniques:

a) 87675 – 34564
b) 3000121 – 1200123

4.- Calculate with three different algorithmic techniques. Explain and justify the three proposed techniques:

a) 2345 × 789
b) 2900150007 × 93500680

5.- Calculate the quotient and the difference in the following case, using four techniques and justifying them:

a) Divide 81207 ÷ 75
b) Divide 7300 897 ÷ 365

6.- a) Compile the different criteria of divisibility and try to translate them into the conditions of the additive and hybrid NS.
b) Calculate the common divisors and the common multiples of 24 y 30, and of 372 y 222.
c) Calculate the highest common divisor and the least common multiple of 24 and 30, and of 372 y 222.

Questions about the decimal positional NS

- Calculate 3001005 – 2890719
- Calculate 630098 × 700400
- Divide 74800358 ÷ 379
- Find the greatest common divisor and the least common multiple of (735 y 2970)

a) Can all the numbers be written in the positional NS?
b) What symbols are used in the positional NS and what kind of symbols are they? How are the different powers of the base represented? What does its being positional mean?
c) Can the different arithmetic operations always be done? With what numbers are the calculations almost impracticable (domain of validity)?
d) In the positional NS are the adding and multiplication tables necessary? Why? What are these tables like?
e) At what point does the magnitude of numbers make the operations difficult to perform without any error? (reliability)
f) Which techniques are used to operate quickly?
g) How is the written number modified when it is multiplied by a power of the basis?
h) What changes are there in the decimal positional NS with regard to the additive NS (Egyptian and Roman), to the hybrid NS (Chinese and oral) and to the "no decimal positional" or primitive NS?

Materials for the 10^{th} session (pp. 28 and 32)

Summary of the study of the numeral systems

A useful written representation of the natural numbers has to be at least as follows:

(A) The writing has to be univocal and easy to use. Few symbols, easy to memorize them and the length of the writings has to make its reading easy.
(B) The comparison of numbers from its writings has to be as easy as possible.
(C) The calculations have to be done in a rapid, simple and trustworthy way.

Three kinds of numeral systems:

(1) **I: "additive" numeral systems**

- In a first moment, numbers for 1, n, n^2, n^3, n^4, n^5... where n is the ***base of the numeration system***. *Example: Egyptian system*: 1, *10*, 10^2, 10^3, 10^4, 10^5 ...
- In a second moment, numbers for 1, a, n, an, n^2, an^2, n^3, an^3, n^4..., and $a < n$ is a privileged divisor that receives the name of ***auxiliary base of the system***. *Example: Roman system:* 1, 5, 10, 5×10, 10^2, 5×10^2, 10^3...

(2) **II: "hybrid" numeral systems**

Two kinds of symbols:

- Symbols that represent the different powers of the base n, that is, n^0, n, n^2, n^3, n^4, ...
- Symbols that represent the multipliers of the above mentioned powers, that is, for 2, 3,..., (n–1) (they are the coefficients).

They always have to be placed above or in front of the power to which they multiply (here the position is important).
Example: the *Chinese- Japanese system* and our oral decimal system.

(3) **III: "positional" numeral system**

- Only numbers to represent the first natural numbers $n-1$: 1, 2, 3,..., ($n-1$) (they work as coefficients of the powers of the base n).
- Each one of the powers n^i is represented by the position "i^{th}" that a coefficient occupies within the group of symbols that represent the number.

Example: *Mayan and Babylonian system* (primitive with symbols for 1 and *a, a privileged divisor of n and for zero*). Written decimal numeral system.

A numeral system must allow us to express natural numbers with a set of symbols **S**, in such a way that the following conditions are compatible:

(1) There is no ambiguity.
(2) The cardinal of the set of symbols **S used** is not excessively big.
(3) Every natural number is represented by a chain of symbols (repeated or not) that is not excessively long.
(4) The techniques are powerful, economic and trustworthy to:
    (4.1) compare two or more numbers
    (4.2) add two or more numbers
    (4.3) subtract a number from a bigger one
    (4.4) multiply two numbers

(4.5) divide one number by another one
(4.6) find multiples and divisors of one or more numbers

Extreme techniques

- Use only one symbol and repeat it as many times as the number that we want to represent indicates. . This technique allows us to solve (1) y (2) but not (3).
- Use a different symbol for each natural number. This technique allows us to solve (1) y (3) but not (2).

The first intermediate technique

It consists of improving the technique of only one symbol by doing only one type of grouping:

IIIII IIIII IIIII IIIII III ⟶ **23**

V V V V III ⟶ **23**

A new symbol appears for the grouping.

The "additive" numeral system

- It consists of doing groups in a regular form, it uses groupings of the first order, of the second order, etc.
- Similar to the *Egyptian system*.
- Symbols: I, A, B , C , D , E y F

↓ ↓ ↓ ↓ ↓ ↓ ↓

10^0, 10^1, 10^2, 10^3, 10^4, 10^5 y 10^6

They designate successive units, each of which is ten times the previous one.

Egyptian numeral system

- The symbols represent the powers of the base.
- The base is 10. The groupings are in tens.
- Ambiguity of writing does not exist.
- The addition is used.
- It is not positional. It does not take zero as a number.
- It is not possible to write any number.

Characteristics of additive SN

- It does not present ambiguities.
- There are still difficulties to solve the task of diminishing the number of necessary symbols to represent every number. We need 63 symbols to represent 9.999.999.
- It does not allow to compare two numbers in a satisfactory way.

- It allows to do the tasks of adding up, subtracting, multiplying and dividing for small numbers in a "reasonably economical way". But, if the numbers increase, the technique is not very economical.
- It presents the advantage of not having to resort to using the adding and multiplying tables.

A direction of evolution of the **additive representation technique**

Aim: Give a **more effective and economical** response to the task of diminishing the necessary number of symbols to represent every number.

Variation contributed by the *Roman system*:

Symbols: I, V, X, L, C, D y M

↓ ↓ ↓ ↓ ↓ ↓ ↓

10^0, 5×10^0, 10^1, 5×10^1, 10^2, 5×10^2 y 10^3

They designate successive units that are alternatively the quintuple or the double of the previous one.

Another variation inside the Roman system

Every symbol that is placed on the left of another symbol which has the immediate higher value, shows that the smaller number has to be subtracted from the bigger one.

9 → IX; 4 → IV; 40 → XL; 90 → XC;

400 → CD; 900 → CM.

These improvements, to give response to the task of diminishing the number of necessary symbols to represent every number, make this technique **less economical** and **less effective** to do arithmetic operations, even with small numbers.

CONCLUSION

THE MAIN AIM OR "RAISON D'ÊTRE" of the *additive* numeral systems is the *representation of the natural numbers* without ambiguities and with a small quantity of symbols and not the simplification of algorithms of the arithmetical operations.

Hybrid numeral system

- It is an **additive-multiplicative** system, similar to the *Chinese* or to our *oral system*.
- It uses the symbols for the powers of the base:

  I → 10^0; A → 10^1; B → 10^2; C → 10^3;

  D → 10^4; E → 10^5; F → 10^6; ...
- It uses a new type of symbols with the function of **multipliers** of the above mentioned powers (2, 3, 4, 5, 6, 7, 8 y 9).
- It improves the solution to the tasks of diminishing the necessary number of symbols to represent every number, since it **avoids the repetition of symbols of the powers** of the base.
- The representation of the numbers is simplified, though still we cannot write all the natural numbers with a finite quantity of symbols, fixed in advance.

- The **algorithm for the comparison** of numbers is **more economical** than in the additive NS, since it is **not necessary to count** how many times every symbol is repeated.
- The **arithmetic operations** can be done in an **easier and more economical** way than in the additive NS, especially with **"big" numbers**.
- Nevertheless, it is **necessary the adding of coefficients table and the two tables of multiplying** (the one of the powers of the base and the other one of the coefficients).

Chinese system

- It has symbols that represent the powers of the base and its multipliers.
- The base is 10. It groups in tens.
- There is no ambiguity of writing.
- The position is important. It does not take zero as a number.
- It uses the addition and multiplication operations.
- It is not possible to write any number.

Characteristics of positional NS

- It gives a definitive response to (2), and **allows to solve the comparison and calculation tasks**.
- There are **no symbols for the powers** of the base since these are indicated by the different positions where the respective coefficients are placed.
- It allows to **write all the natural numbers with a finite set of symbols** (fixed in advance).
- The only used symbols are the coefficients or multipliers of the powers of the base.

Characteristics of the primitive positional NSystems

- They use symbols for the coefficients that try to evoke visually what they represent.

  Examples: **9 → I •••• , 87 → •••• I••**

  57 → 〈〈〈〈〈 ∨ ∨ ∨ ∨ ∨ ∨ ∨ , **247 →** ∨ ∨ ∨ ∨   ∨ ∨ ∨ ∨ ∨ ∨ ∨
- It does not allow to solve the task "How to express the natural numbers that we need by means of symbols so that there is no ambiguity?"

  III can be "15" or "110" ó "205".

  ∨ ∨ **can be "2" or "61".**

The "decimal positional" NS

- It gives a definitive response to the task of diminishing the number of necessary symbols to represent every number.
- The symbols of the powers of the base are not necessary, since these are indicated by the position of the respective coefficients.
- It allows to write all the numbers with a finite number of symbols (fixed in advance).

- The only symbols are the coefficients or multipliers of the powers of the base. The powers of the base are indicated by their position.
- It uses different conventional symbols for each of the coefficients (smaller than the base numbers).
- It adds a new symbol: **zero**, to indicate that in a specific position there is **absence of elements**.
- The base is 10 and has the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9 for the coefficients and the symbol 0 to indicate the absence of elements in a specific position.
- It does not present ambiguities.
- It allows to compare two numbers in a very effective way.
- It provides a more economical and effective response to the calculation tasks.

Initial question: What special properties and characteristics does our NS (decimal positional) have that it is the only one used in almost every country, thus being imposed itself on all the others NS that have existed in the history and coexisted for many centuries?

Our decimal positional numeral system (SN_p) must have special properties that allow it to express the cardinal of a finite collection (that is, a natural number),no matter how very big that it should be and to handle the numbers comparing them, effecting calculations and constructing other numbers from the known ones. Which are these special properties?

Materials for the 11th session (p. 33)

Individual evaluation of the students

IES San Juan Bautista
Evaluation 3rd E.S.O.

Name:..

1.- Do the following **calculations** in the **additive** and in the **hybrid numeral system**, emphasizing the principal limitations and advantages of the adopted NS in every case**:**

- **4001 – 2015**
- **Divide 6407 ÷ 372** (4p)

2.- Construct a **positional numeral** system with **4 symbols** and **base 60**, and write the numbers 3, 11, 30, 60, 100, 3661 y 216216 in that system. **Explain** the **limitations** and **advantages** of this system. Can the writings present some case of **ambiguity**? If that is the case, **give an example** and indicate how this problem of ambiguity might be avoided. (2p)

3.- Juan has made a **mistake** in the following **calculation**:

$$\begin{array}{r} 387 \\ \times\ 48 \\ \hline 3096 \\ 1548 \\ \hline 4644 \end{array}$$

What is the connection **between 4644** and **387**? **Justify** your answer. (1p)

4.- The Romans used **two techniques to represent** numbers. The following numbers are written using the later technique: 4 –> IV, 49 –> XLIX , 99 –> XCIX, 499 –> CDXCIX, 999 –> CMXCIX

Characterize these two **techniques to represent** the numbers**,** indicating **their advantages and limitations**. (2p)

5.- Knowing that 15368 × 18 = 276624 and that 15368 × 23 = 353464.

Do the following calculations, without doing multiplicative calculations (you can use the previous calculations and the properties of the operations):

23018 × 15368, **41 × 15368**.

Justify the response, indicating the used properties. (1p)